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We show that the gauge orbits of non-Abelian embedded monopoles are associated with a representation of the residual gauge symmetry. This is consistent with non-Abelian monopoles lying within multiplets that form representations of the residual gauge symmetry. Implications of this result are discussed in detail.

I. INTRODUCTION

Gauge field theories provide the modern framework for describing the interactions between particles, with their nature prescribed by the relevant symmetries of the situation. A remarkable feature of such interactions is that, in addition to describing the force between particles, the structure of the symmetries can also provide stable configurations of the constituent fields. An example of such configurations are monopoles, which are charged finite energy solutions localised in space. Consequently, a relevant and natural question to ask is how do these monopoles interact with each other?

This question has a history dating back to soon after 't Hooft and Polyakov's description of monopoles in spontaneously broken gauge theories [1]. To answer such a question requires a definite characterisation of the symmetries of the monopole configuration. Symmetries that relate to the residual gauge group H are local and as such can be expected to give rise to a gauge interaction. The form of this interaction should naturally be prescribed by the relevant symmetries of the monopole configuration.

A specific symmetry of non-Abelian monopoles has been discovered by Goddard, Nuyts and Olive [2]. Following Englert and Windey [3], they considered the spectrum of magnetic charges Q of non-Abelian monopoles to be defined by the monopole's asymptotic magnetic field

$$\boldsymbol{B} \sim \frac{\hat{\boldsymbol{r}}}{4\pi r^2} Q. \tag{1}$$

Topological arguments then lead to the quantisation of the magnetic charge [2,3]

$$\exp(eQ) = 1. \tag{2}$$

The general solution to which is that any magnetic charge is gauge equivalent to one of the form

$$Q = \frac{4\pi}{e} \sum_{a=1}^{l} n_a \boldsymbol{\beta}^*_{(a)} \cdot \boldsymbol{T} \in \mathcal{H}. \tag{3}$$

Here the T_i are a set of mutually commuting generators of the residual symmetry group, and $\boldsymbol{\beta}^*_{(a)} = \boldsymbol{\beta}_{(a)}/{\boldsymbol{\beta}_{(a)}}^2$ are the duals to l simple roots. The integers n_a label the specific monopole. Generally these are not gauge invariant and transform into one another under the Weyl subgroup of the residual symmetry. This group is described by roots $\boldsymbol{\alpha} \in \Phi(H)$ and transforms the associated dual root to its reflection in the hyperplane $\boldsymbol{\alpha} \cdot \boldsymbol{x} = 0$.

Goddard, Nuyts and Olive interpreted the lattice of magnetic charges in Eq. (3) as being the weights of a dual group H^* such that

$$\frac{eQ}{4\pi} \in \Lambda(H^*). \tag{4}$$

Given this interpretation they proposed two alternative implications for their results: (a) the weights label the different monopoles; or, more strongly, (b) the weights are labelling the representation of the monopole. In case (b) this means that the dual group describes the symmetries of the monopole configuration and would then be the natural candidate for the gauge group describing the interaction of monopoles.

In this paper we address the symmetry of non-Abelian monopoles from a slightly different perspective. In particular we consider 't Hooft-Polyakov monopoles embedded within a larger gauge theory. The importance of such configurations is that their spectrum may be described algebraically by embedding an $su(2)$ algebra

$$\begin{array}{ccc} \mathcal{G} & \rightarrow & \mathcal{H} \\ \cup & & \cup \\ su(2)_Q & \rightarrow & u(1)_Q. \end{array} \tag{5}$$

We shall exploit this algebraic structure to determine the different possible embeddings of the associated monopole, and hence determine directly the symmetries of particular non-Abelian monopole configurations.

Our method is to describe a monopole degeneracy by an associated gauge orbit, which is a collection of rigidly gauge equivalent monopoles. This gauge orbit is exactly analogous to associating an orbit $H \cdot \Psi$ with a particle Ψ that transforms under a gauge symmetry H. As such it is representative of both the gauge symmetry and the representation that the particle transforms under. Our main result is that the orbit of rigidly gauge equivalent embedded monopoles has the isomorphism

$$\mathcal{O} \cong \mathrm{Ad}(H)X, \tag{6}$$

where X is an associated element of $\mathcal{M}$, the set of all generators of massive gauge bosons. This appears to give a correspondence between the transformations of magnetic monopoles and the transformations of the massive gauge bosons under the residual symmetry group H.

Given this method, the plan of this paper is as follows. Firstly we introduce the framework of embedded monopoles, relating them to fundamental monopoles and describing the associated algebraic structure. This allows us to discuss the gauge transformations of embedded monopoles and we utilise the associated algebraic structure to describe the gauge orbits, proving Eq. (6), the main result of this work. Then, to provide a specific example, we illustrate many of our results with Georgi-Glashow $SU(5) \to SU(3) \times SU(2) \times U(1)/\mathbf{Z}_6$ symmetry breaking. Finally we draw our conclusions and discuss the implications of this work.

II. EMBEDDED AND FUNDAMENTAL MONOPOLES

A. Framework

We consider a spontaneously broken Yang-Mills theory defined by a simple, compact gauge group G and a scalar field Φ in the adjoint representation of G. The dynamics of the interacting scalar and gauge fields are described through the following Lagrangian

$$\mathcal{L}[\Phi, A^\mu] = -\tfrac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu}\rangle + \tfrac{1}{2}\langle D_\mu\Phi, D^\mu\Phi\rangle - V[\Phi], \quad (7)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu A_\nu], \quad (8)$$
$$D_\mu\Phi = \partial_\mu\Phi + e[A_\mu, \Phi], \quad (9)$$

and the inner product is taken from the Killing form

$$\langle X, Y\rangle = -2\,\mathrm{tr}[\mathrm{ad}(X)\mathrm{ad}(Y)], \quad (10)$$

representing the natural $\mathrm{Ad}(G)$-invariant inner product on the algebra $\mathcal{G}$.

Taking the potential $V[\Phi]$ to be minimised by the scalar field value Φ_0, the residual symmetry group consists of elements $h \in H$ satisfying

$$\mathrm{Ad}(h)\Phi_0 = \Phi_0. \quad (11)$$

Then the inclusion of $H \subset G$ defines a relevant decomposition of $\mathcal{G}$ into a set of massless and a set of massive gauge boson generators

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}. \quad (12)$$

This decomposition is respected by the adjoint action of the residual symmetry H

$$\mathrm{Ad}(H)\mathcal{H} \subseteq \mathcal{H}, \qquad \mathrm{Ad}(H)\mathcal{M} \subseteq \mathcal{M}, \quad (13)$$

so that the massive gauge bosons form a representation of H. In general this representation may be reducible, and decomposes into irreducible representations

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n. \quad (14)$$

Each set corresponding to a gauge family of massive gauge bosons.

Since we are considering the scalar field to lie in the adjoint representation of H then necessarily

$$\mathrm{rank}(\mathcal{G}) = \mathrm{rank}(\mathcal{H}) = l. \quad (15)$$

Hence we may choose a common Cartan subalgebra $\mathcal{T}$ for both $\mathcal{H}$ and $\mathcal{G}$. Then the vacuum may be denoted

$$\Phi_0 = v\,\boldsymbol{h}\cdot\boldsymbol{T}, \quad (16)$$

with respect to a suitable orthonormal basis $\{T_1, \cdots, T_l\}$. This is guaranteed by the following result [6]: *Given a Cartan subalgebra $\mathcal{T} \subseteq \mathcal{H}$, for any element $X \in \mathcal{H}$ there exists an $h \in H$ such that $\mathrm{Ad}(h)X \in \mathcal{T}$.*

B. 't Hooft-Polyakov Monopoles

We shall start by quickly reviewing 't Hooft-Polyakov monopoles [1]. These occur in the spontaneously broken gauge theory

$$su(2)_Q \to u(1)_Q. \quad (17)$$

Here Q is a label that will become apparent later. To describe the monopole solution we shall usefully split $su(2)_Q$ as in Eq. (12)

$$su(2)_Q = u(1)_Q \oplus \mathcal{M}_Q. \quad (18)$$

Then we may further write

$$\mathcal{M}_Q = \mathbf{R}\,t^1_Q \oplus \mathbf{R}\,t^2_Q, \quad (19)$$
$$u(1)_Q = \mathbf{R}\,t^3_Q, \quad (20)$$

where $\{t^1_Q, t^2_Q, t^3_Q\}$ generate $su(2)_Q$ and are orthonormal with respect to Eq. (10)

$$\langle t^i_Q, t^j_Q\rangle = \delta_{ij}. \quad (21)$$

It should be noted that we have implicitly taken the decomposition in Eqs. (19,20) with respect to the vacuum $\Phi_0 = vt^3$.

Given this structure, the 't Hooft-Polyakov monopole solution can be written in the form

$$\Phi(\boldsymbol{r}) = vh(r)\,\mathrm{Ad}(g(\Omega))\Phi_0, \quad (22)$$
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1-u(r)}{er}\,\hat{\boldsymbol{r}} \wedge \boldsymbol{t}_Q, \quad (23)$$

with a scalar field angular structure described by

$$g(\Omega) = \exp(\theta(t_Q^2 \sin\varphi - t_Q^1 \cos\varphi)). \tag{24}$$

Note that $\mathrm{Ad}(g(\Omega))\Phi_0 = v\,\hat{\boldsymbol{r}}\cdot\boldsymbol{t}_Q$ gives the usual 't Hooft-Polyakov expression.

Asymptotically the profile functions behave as

$$h(r) \sim 1, \quad u(r) \sim 0, \tag{25}$$

from which the asymptotic magnetic field is found from the space-space components of the gauge field tensor

$$\boldsymbol{B}(\boldsymbol{r}) \sim \frac{\hat{\boldsymbol{r}}}{er^2}\cdot\boldsymbol{t}_Q = \frac{\hat{\boldsymbol{r}}}{4\pi r^2}\cdot\frac{4\pi}{e}\boldsymbol{t}_Q; \tag{26}$$

a magnetic monopole. From this solution we extract the magnetic charge by considering the gauge orientation of the magnetic field in the unitary gauge

$$Q = \frac{4\pi}{e} t_Q^3, \tag{27}$$

as in Eq. (1). Equivalently this may be considered as the asymptotic magnetic field in the direction $\hat{\boldsymbol{r}}^3$, where the scalar field tends to Φ_0.

C. Embedded Monopoles

Embedded monopoles [7] are of the 't Hooft-Polyakov form, but contained within a larger gauge theory. Then both the scalar and gauge fields take non-zero values only on an $su(2)_Q \to u(1)_Q$ subtheory. This is consistently described by embedding the $su(2)_Q$ gauge algebra in the following manner

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{H} \\ \cup & & \cup \\ su(2)_Q & \to & u(1)_Q. \end{array} \tag{28}$$

Then the magnetic charge $Q \in \mathcal{H}$ and vacuum Φ_0 are elements of the full theory.

The aim of this section is to find the embeddings in Eq. (28) that give rise to embedded monopole solutions. Not all embeddings do, and it is only when the above 't Hooft-Polyakov type Ansatz (22,23) satisfies the field equations that a solution is admitted.

Recalling Eqs. (12,18), it is clear that the monopole embedding can be described in terms of its algebraic structure

$$\begin{array}{ccccc} \mathcal{G} & = & \mathcal{H} & \oplus & \mathcal{M} \\ \cup & & \cup & & \cup \\ su(2)_Q & = & u(1)_Q & \oplus & \mathcal{M}_Q. \end{array} \tag{29}$$

The preservation of $\mathcal{M}$ under the action of H implies that the relative decompositions in Eq. (29) are similarly preserved under the action of $h \in H$

$$su(2)_Q \to \mathrm{Ad}(h) su(2)_Q. \tag{30}$$

This property transpires to be of crucial importance later in this paper.

Our result classifying the embedded monopole spectrum is: the Ansatz of Eqs. (22,23) defines a solution only for an $su(2)_Q$ embedding with

$$\mathcal{M}_Q \subseteq \mathcal{M}_i. \tag{31}$$

Thus $\mathcal{M}_Q$ is a subspace of one of the gauge families $\mathcal{M}_i$ in Eq. (14). This constitutes the central result of this section.

Although we shall provide a proof of Eq. (31) in Appendix A, we now show that it relates to an analogous result on the classification of embedded vortices. These are Nielsen-Olesen vortices embedded in a larger symmetry breaking [7]

$$\begin{array}{ccc} G & \to & H \\ \cup & & \cup \\ U(1)_X & \to & \mathbf{1}, \end{array} \tag{32}$$

where $U(1)_X = \exp(\mathbf{R}X)$. Their classification states that an embedded vortex constitutes a solution to the equations of motion providing that [8]

$$X \in \mathcal{M}_i. \tag{33}$$

The point is that the spectrum of embedded vortices is related to the spectrum of embedded monopoles through the following tower of embeddings

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{H} \\ \cup & & \cup \\ su(2)_Q & \to & u(1)_Q \\ \cup & & \cup \\ u(1)_X & \to & \mathbf{0}. \end{array} \tag{34}$$

From this we conclude that every $X \in \mathcal{M}_Q$ defines an embedded vortex. Hence, by the result in Eq. (33)

$$\mathcal{M}_Q \subseteq \mathcal{M}_i. \tag{35}$$

This shows that the monopole embedding may only be of the form in Eq. (31). However, to prove that every such embedding produces a solution is more difficult, requiring a direct examination of the field equations. Such a proof is provided in Appendix A.

D. Fundamental Monopoles

Fundamental monopoles [4] constitute an important subset of embedded monopoles. They are a vital concept because any monopole solution, embedded or otherwise, may be interpreted as a composition of them.

In particular, as discussed in the introduction, a general solution of the quantisation condition $\exp(eQ) = 1$ is of the form

$$Q = \frac{4\pi}{e} \sum_{a=1}^{l} n_a \boldsymbol{\beta}^*_{(a)} \cdot \boldsymbol{T} \in \mathcal{H}. \tag{36}$$

Here $\boldsymbol{\beta}^*_{(a)}$ are the duals to l simple roots

$$\boldsymbol{\beta}^*_{(a)} = \frac{1}{\beta^2_{(a)}} \boldsymbol{\beta}_{(a)}. \tag{37}$$

The monopoles associated with these simple roots are the fundamental monopoles.

In fact this set of fundamental monopoles splits into massive and massless varieties [5], with the massless monopoles associated with the roots of H. We shall discuss here only the massive fundamental monopoles, which correspond to the roots of G that are not roots of H. Furthermore, we shall refer to the set of all possible massive fundamental monopoles.

It will be necessary to discuss the roots $\Phi(\mathcal{G})$ and the root structure of G in more detail. Recall that a root $\boldsymbol{\alpha}$, and its associated root space E_α, are defined by

$$i\,\mathrm{ad}(\boldsymbol{T}) E_\alpha = \boldsymbol{\alpha} E_\alpha. \tag{38}$$

These E_α's may be conveniently normalised to

$$[E_\alpha, E_{-\alpha}] = i\boldsymbol{\alpha} \cdot \boldsymbol{T}. \tag{39}$$

The important consideration here is that associated with each root is an $su(2)$ algebra with an orthonormal basis

$$\begin{aligned} t^1_\alpha &= (2\alpha^2)^{-\frac{1}{2}}(E_\alpha + E_{-\alpha}), \\ t^2_\alpha &= -i(2\alpha^2)^{-\frac{1}{2}}(E_\alpha - E_{-\alpha}), \\ t^3_\alpha &= \boldsymbol{\alpha}^* \cdot \boldsymbol{T}. \end{aligned} \tag{40}$$

Denoting the associated algebra by $su(2)_\alpha$, it is clear that when $\boldsymbol{\alpha} \in \Phi(\mathcal{H})$ then $su(2)_\alpha \subseteq \mathcal{H}$.

The suggestion made by Eq. (40) is that massive fundamental monopoles are embedded monopoles. In keeping with this we associate with Eq. (18)

$$su(2)_\beta = u(1)_\beta \oplus \mathcal{M}_\beta, \tag{41}$$

with the identification

$$u(1)_\beta = \mathbf{R}\, \boldsymbol{\beta} \cdot \boldsymbol{T}, \tag{42}$$

$$\mathcal{M}_\beta = \mathbf{R}\, t^1_\beta \oplus \mathbf{R}\, t^2_\beta, \tag{43}$$

relating to Eqs. (19,20).

Before proving that the $su(2)_\beta$ embedding associated with a massive fundamental monopole does indeed define an embedded monopole we shall firstly explore some features of the above definitions. It transpires that the algebraic features of Eqs. (42,43) allow us to prove that the above structure implies an associated embedded monopole.

The main algebraic features of Eqs. (42,43) arises from considering the collection of all such algebras

$$\mathcal{H} = \mathcal{T} \oplus \sum_{\alpha \in \Phi(\mathcal{H})} \mathcal{M}_\alpha, \tag{44}$$

$$\mathcal{G} = \mathcal{T} \oplus \sum_{\gamma \in \Phi(\mathcal{G})} \mathcal{M}_\gamma. \tag{45}$$

Then, recalling $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$, one has an associated decomposition

$$\Phi(\mathcal{G}) = \Phi(\mathcal{H}) + \Phi(\mathcal{M}), \tag{46}$$

with the roots in $\mathcal{M}$ defining

$$\mathcal{M} = \sum_{\beta \in \Phi(\mathcal{M})} \mathcal{M}_\beta. \tag{47}$$

The main mathematical observation of this section is that associated with the decomposition of $\mathcal{M}$ into the gauge families of Eq. (14)

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n, \tag{48}$$

is a corresponding decomposition of $\Phi(\mathcal{M})$

$$\Phi(\mathcal{M}) = \Phi(\mathcal{M}_1) + \cdots + \Phi(\mathcal{M}_n), \tag{49}$$

where each

$$\mathcal{M}_i = \sum_{\beta \in \Phi(\mathcal{M}_i)} \mathcal{M}_\beta. \tag{50}$$

We shall delay proof of Eqs. (49,50) to the end of this section, and firstly discuss the consequences for monopole classification.

The consequence of the result in Eq. (50) is that all massive fundamental monopoles are embedded, since trivially $\mathcal{M}_\beta \subset \mathcal{M}_i$. Therefore fundamental monopoles correspond to the roots of $\Phi(\mathcal{M})$, and such roots define subspaces $\mathcal{M}_\beta$ associated with the monopole embeddings. Mathematically, each magnetic charge $Q = \boldsymbol{\beta}^* \cdot \boldsymbol{T}$ is defined by a root $\boldsymbol{\beta} \in \Phi(\mathcal{M})$ that specifies an $su(2)$ embedding

$$su(2)_Q = su(2)_\beta. \tag{51}$$

This embedding defines a reduction into

$$u(1)_Q = u(1)_\beta \subseteq \mathcal{H}, \tag{52}$$

$$\mathcal{M}_Q = \mathcal{M}_\beta \subseteq \mathcal{M}, \tag{53}$$

such that each $\mathcal{M}_i$ has the decomposition into subspaces corresponding to distinct fundamental monopoles

$$\mathcal{M}_i = \sum_{\beta \in \Phi(\mathcal{M}_i)} \mathcal{M}_\beta. \tag{54}$$

We now discuss proof of Eqs. (49,50). Consider the decomposition of $\mathcal{M}$ into the gauge families $\mathcal{M}_i$. Then the adjoint action of $\mathrm{Ad}(\exp(\mathcal{T})) \subseteq \mathrm{Ad}(H)$ subdivides each $\mathcal{M}_i$ further into

$$\mathcal{M}_i = \mathcal{M}_i^1 \oplus \cdots \oplus \mathcal{M}_i^k. \tag{55}$$

Now, by considering the power series expansion and using the definition in Eq. (38), we also have

$$\mathrm{Ad}(\exp(\boldsymbol{\theta} \cdot \boldsymbol{T}))E_\beta = \exp(-i\boldsymbol{\beta} \cdot \boldsymbol{\theta})E_\beta. \tag{56}$$

Therefore if we consider the corresponding action on t_β^1 and t_β^2,

$$\mathrm{Ad}(\exp(\boldsymbol{\theta} \cdot \boldsymbol{T})) \begin{pmatrix} t_\beta^1 \\ t_\beta^2 \end{pmatrix} = R(\boldsymbol{\beta} \cdot \boldsymbol{\theta}) \begin{pmatrix} t_\beta^1 \\ t_\beta^2 \end{pmatrix}, \tag{57}$$

with $R(\boldsymbol{\beta} \cdot \boldsymbol{\theta})$ an $SO(2)$ rotation matrix rotating t_β^1 and t_β^2 within their corresponding $\mathcal{M}_\beta$,

$$R(\boldsymbol{\beta} \cdot \boldsymbol{\theta}) = \begin{pmatrix} \cos(\boldsymbol{\beta} \cdot \boldsymbol{\theta}) & \sin(\boldsymbol{\beta} \cdot \boldsymbol{\theta}) \\ -\sin(\boldsymbol{\beta} \cdot \boldsymbol{\theta}) & \cos(\boldsymbol{\beta} \cdot \boldsymbol{\theta}) \end{pmatrix}. \tag{58}$$

From this we conclude that the irreducible subspaces of $\mathrm{Ad}(T)$ acting on $\mathcal{M}$ are precisely the $\mathcal{M}_\beta$. Consequently the $\mathcal{M}_i^j$ of Eq. (55) are identified with these $\mathcal{M}_\beta$. This completes the proof.

E. Embedded Combinations of Fundamental Monopoles

In this section we enquire as to the form of embedded monopoles that are not fundamental. In particular, we discuss a specific set of non-fundamental embedded monopoles.

Non-fundamental embedded monopoles correspond to combinations of fundamental monopoles. The nature of such combinations may be seen by constructing the following embedding

$$\begin{array}{ccc} \mathcal{H} \oplus \mathcal{M}_i & \rightarrow & \mathcal{H} \\ \cup & & \cup \\ su(2)_{\beta_{(1)}} \times \cdots \times su(2)_{\beta_{(p)}} & \rightarrow & u(1)_{\beta_{(1)}} \times \cdots \times u(1)_{\beta_{(p)}} \end{array} \tag{59}$$

such that the set $\{\boldsymbol{\beta}_{(1)}, \cdots, \boldsymbol{\beta}_{(p)}\} \in \Phi(\mathcal{M}_i)$ are mutually orthogonal roots, in that

$$\boldsymbol{\beta}_{(i)} \cdot \boldsymbol{\beta}_{(j)} = 0, \quad i \neq j. \tag{60}$$

In such a case the root spaces $E_{\beta_{(i)}}$ and $E_{\beta_{(j)}}$ commute. Hence Eq. (40) implies that $[\mathcal{M}_{\beta_{(i)}}, \mathcal{M}_{\beta_{(j)}}] = 0$, a necessary requirement for the structure in Eq. (59).

Physically the embedding in Eq. (59) corresponds to identifying a maximal set of fundamental monopoles whose gauge fields commute with one another.

The point of this embedding that it is possible to directly construct diagonal $su(2)$ algebras lying between the other $su(2)$ algebras in the above embedding. Such diagonal $su(2)$ subalgebras may also define embedded monopoles. The existence of these is found by explicit construction, such that they have a basis

$$t^1 = n_1 t^1_{\beta_{(1)}} + \cdots + n_p t^1_{\beta_{(p)}}, \tag{61}$$
$$t^2 = n_1 t^2_{\beta_{(1)}} + \cdots + n_p t^2_{\beta_{(p)}}, \tag{62}$$
$$t^3 = (n_1 \tfrac{1}{\beta^2_{(1)}} \boldsymbol{\beta}_{(1)} + \cdots + n_p \tfrac{1}{\beta^2_{(p)}} \boldsymbol{\beta}_{(p)}) \cdot \boldsymbol{T}, \tag{63}$$

where all $n_i \in \{0, 1\}$. It may be easily checked that they define an $su(2)$ algebra.

III. ORBIT STRUCTURE OF EMBEDDED MONOPOLES

A. Gauge Transformations of Embedded Monopoles

A rigid gauge transformation by an element $h \in H$ acts upon the scalar and gauge fields of an embedded monopole by transforming

$$\Phi(\boldsymbol{r}) \mapsto \mathrm{Ad}(h)\Phi(\boldsymbol{r}), \tag{64}$$
$$\boldsymbol{A}(\boldsymbol{r}) \mapsto \mathrm{Ad}(h)\boldsymbol{A}(\boldsymbol{r}). \tag{65}$$

Recalling the embedded monopole Ansatz in Eqs. (22,23)

$$\Phi(\boldsymbol{r}) = vh(r)\,\mathrm{Ad}(g(\Omega))\Phi_0, \tag{66}$$
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1-u(r)}{er}\,\hat{\boldsymbol{r}} \wedge \boldsymbol{t}_Q, \tag{67}$$
$$g(\Omega) = \exp(\theta(t_Q^2 \sin\varphi - t_Q^1 \cos\varphi)), \tag{68}$$

it is clear that the transformation in Eqs. (64,65) is entirely equivalent to transforming

$$\boldsymbol{t}_Q \mapsto \mathrm{Ad}(h)\boldsymbol{t}_Q. \tag{69}$$

This relates to the embedding of $su(2)_Q = u(1)_Q \oplus \mathcal{M}_Q$ in the following way

$$Q \mapsto \mathrm{Ad}(h)Q, \quad \mathcal{M}_Q \mapsto \mathrm{Ad}(h)\mathcal{M}_Q. \tag{70}$$

In conclusion a rigid gauge transformation of the embedded monopole may be described through a transformation of its magnetic charge *and* its associated $\mathcal{M}_Q$.

We start by firstly considering those rigid gauge transformations that are defined by elements

$$h(\chi) = \exp(Q\chi) \in U(1)_Q, \tag{71}$$

so that the corresponding action is trivial upon $u(1)_Q$. These transform $\mathcal{M}_Q$ to itself, representing an internal transformation of the $SU(2)_Q$ embedding. Explicitly

$$\begin{pmatrix} t_Q^1 \\ t_Q^2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} t_Q^1 \\ t_Q^2 \end{pmatrix}, \tag{72}$$

which is entirely equivalent to transforming

$$g(\theta,\varphi) \mapsto g(\theta,\varphi+\chi). \tag{73}$$

Consequently the effect of a rigid gauge transformation defined by $h(\chi)$ upon the monopole solution is to rotate

$$\Phi(r,\theta,\varphi) \mapsto \Phi(r,\theta,\varphi+\chi), \tag{74}$$
$$\boldsymbol{A}(r,\theta,\varphi) \mapsto \boldsymbol{A}(r,\theta,\varphi+\chi). \tag{75}$$

Rigidly rotating the monopole around the $(\theta,\varphi)=(0,0)$ axis. This is specified by the embedding of $H \subset G$ such that the action of $h(\chi)$ keeps $\Phi_0 = \Phi(\infty,0,0)$ invariant.

Next we consider the effects of a gauge transformation on the magnetic charge Q. By Eq. (80) the magnetic charge transforms under a rigid gauge transformation to

$$Q \mapsto \mathrm{Ad}(h)Q. \tag{76}$$

Considering all possible transformations leads to a manifold of gauge equivalent magnetic charges

$$M(Q) = \mathrm{Ad}(H)Q. \tag{77}$$

Analogously to the usual coset space representation of a vacuum manifold, M_Q may be expressed as an isomorphism to the coset space

$$M(Q) \cong \frac{H}{C(Q)}. \tag{78}$$

Here $C(Q)$ is the centraliser of Q in H such that

$$C(Q) = \{h \in H \ : \ \mathrm{Ad}(h)Q = Q\}. \tag{79}$$

Its algebra consists of elements that commute with Q.

The main point of this section is that all possible rigid gauge transformations of the monopole consist of *both* the rotation by $U(1)_Q$ and the action upon the magnetic charge. This may be consistently described by collectively taking the transformations in Eqs. (72,76) together. Then all rigid gauge transformations of an embedded monopole may be jointly considered as a transformation of its $su(2)$ embedding

$$su(2)_Q \mapsto \mathrm{Ad}(h)su(2)_Q. \tag{80}$$

Our tactic for examining the gauge transformation properties of embedded monopoles is to examine the features of Eq. (80) acting upon the $su(2)$ embeddings of the monopoles.

The gauge equivalent embedded monopoles collectively form a manifold that characterises their gauge equivalence structure

$$M(su(2)_Q) \cong \frac{H}{C(su(2)_Q)}. \tag{81}$$

Here $C(su(2)_Q)$ is the centraliser of $su(2)_Q$ in H,

$$C(su(2)_Q) = \{h \in H \ : \ \mathrm{Ad}(h)su(2)_Q = su(2)_Q\}. \tag{82}$$

The above expression for $M(su(2)_Q)$ describes the orbit of gauge equivalent embedded monopoles. It is completely characteristic of the gauge equivalence structure of these monopoles.

We may convert the above expression into a more useful form by the following result, proved below

$$C(su(2)_Q) = C(X), \quad X \in \mathcal{M}_Q. \tag{83}$$

Hence the gauge orbit of embedded monopoles may be rewritten as

$$M(su(2)_Q) \cong \frac{H}{C(X)}, \quad X \in \mathcal{M}_Q. \tag{84}$$

This constitutes the main result of this section. It equates the gauge orbit of embedded monopoles with an orbit formed by acting the residual symmetry group upon an associated element X.

To prove Eq. (83) we make two statements. The first of which is

$$C(su(2)_Q) = C(\mathcal{M}_Q), \tag{85}$$

where $C(\mathcal{M}_Q)$ is the centraliser of $\mathcal{M}_Q$. This follows immediately from the commutation relations of $su(2)_Q$: since $[X,Y]=Q$ then $C(\mathcal{M}_Q) \subset C(U(1)_Q)$ and Eq. (85) is implied.

The next statement is that

$$C(\mathcal{M}_Q) = C(X), \tag{86}$$

for any non-trivial $X \in \mathcal{M}_Q$. This follows from again using $C(\mathcal{M}_Q) \subset C(U(1)_Q)$, which implies that $U(1)_Q$ commutes with $C(\mathcal{M}_Q)$. Then since any non-trivial $X' \in \mathcal{M}_Q$ is proportional to $\mathrm{Ad}(h)X$ for some $h \in U(1)_Q$ we infer that $C(X') = C(X)$, obtaining Eq. (86).

B. Multiplet Structures of Embedded Monopoles

We now examine the relationship between the multiplet structure of sec. (II D) and the gauge orbit structure of sec. (III A). In particular we relate the decomposition of a gauge family $\mathcal{M}_i$ into its massive fundamental monopole embeddings

$$\mathcal{M}_i = \sum_{\beta \in \Phi(\mathcal{M}_i)} \mathcal{M}_\beta, \tag{87}$$

to the gauge orbit of gauge equivalent monopole embeddings

$$M(su(2)_\beta) \cong \frac{H}{C(X)}, \quad X \in \mathcal{M}_\beta. \tag{88}$$

It seems fairly clear that each of the embeddings $su(2)_\beta$ should correspond to a particular point in the gauge orbit, and indeed this is what we find.

Our technique for examining the multiplet structure of massive fundamental monopoles relate to the approach taken by [2]. They considered transformations taking the magnetic charge back to $\mathcal{T}$,

$$Q \mapsto \mathrm{Ad}(h)Q \in \mathcal{T}, \tag{89}$$

which are defined by the following elements of H

$$w_\alpha = \exp[i\pi(E_\alpha + E_{-\alpha})/\sqrt{2\alpha^2}], \quad \boldsymbol{\alpha} \in \Phi(H). \tag{90}$$

Such elements map

$$Q = \boldsymbol{\beta}\cdot\boldsymbol{T} \mapsto \mathrm{Ad}(w_\alpha)Q = \boldsymbol{\beta}'\cdot\boldsymbol{T}, \tag{91}$$

with $\boldsymbol{\beta}'$ a Weyl reflection of $\boldsymbol{\beta}$ in the hyperplane $\boldsymbol{\alpha}\cdot\boldsymbol{x} = 0$

$$\boldsymbol{\beta} \mapsto \boldsymbol{\beta}' = \sigma_\alpha(\boldsymbol{\beta}) = \boldsymbol{\beta} - 2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}/\boldsymbol{\alpha}^2. \tag{92}$$

All such reflections form a finite group, the Weyl group

$$W = \{\mathbf{1}, w_\alpha : \boldsymbol{\alpha} \in \Phi(H)\} \subset H. \tag{93}$$

We conjecture that the Weyl group is transitive over all roots in each $\Phi(\mathcal{M}_i)$. In that case H is transitive over all magnetic charges in the same gauge family $\mathcal{M}_i$. However, since this result is not central to this paper we shall not consider it further.

We now determine how the Weyl group W acts upon the subspaces $\mathcal{M}_\beta$. To do so observe that when the $su(2)$ algebras containing $u(1)_\beta = \mathbf{R}\boldsymbol{\beta}\cdot\boldsymbol{T}$ are unique then the Weyl reflections can only take

$$\mathcal{M}_\beta \to \mathrm{Ad}(S_\alpha)\mathcal{M}_\beta = \mathcal{M}_{\beta'}. \tag{94}$$

Otherwise the situation is slightly more complicated, as we shall discuss below.

On this issue of when there is a unique such $su(2)$ algebra containing $u(1)_\beta$, consider those algebras of the form

$$su(2) = u(1)_\beta \oplus \mathbf{R}\,t^1 \oplus \mathbf{R}\,t^2, \quad t^1, t^2 \in \mathcal{M}. \tag{95}$$

Writing $E_1 = t_1 + it_2$ and $E_2 = t_1 - it_2$, we use the following algebraic structure

$$i[\boldsymbol{\beta}\cdot\boldsymbol{T}/\beta^2, E_i] = E_i. \tag{96}$$

Expanding both E_1 and E_2 in terms of the root spaces

$$E_1 = \sum_{\gamma\in\Phi(\mathcal{M}_i)} x^1_\gamma E_\gamma, \quad E_2 = \sum_{\gamma\in\Phi(\mathcal{M}_i)} x^2_\gamma E_\gamma, \tag{97}$$

and substituting these into Eq. (96) gives the following coefficients

$$\frac{\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}{\beta^2} x^i_\gamma = x^i_\gamma, \tag{98}$$

i.e. x^i_γ may only be non-zero for $\boldsymbol{\beta}\cdot\boldsymbol{\gamma}/\beta^2 = 1$. Of course this is trivially satisfied for $\boldsymbol{\gamma} = \boldsymbol{\beta}$, and this just yields $su(2)_\beta$ for the associated $su(2)$ algebra. The question is then, when are there other elements of $\Phi(\mathcal{M}_i)$ satisfying $\boldsymbol{\gamma}\cdot\boldsymbol{\beta}/\beta^2 = 0$? This is answered by recalling the Cartan matrix

$$K_{ij} = 2\frac{\boldsymbol{\gamma}_{(i)}\cdot\boldsymbol{\gamma}_{(j)}}{\gamma^2_{(i)}}, \tag{99}$$

associated with a set of simple roots $\{\boldsymbol{\gamma}_{(1)}, \cdots, \boldsymbol{\gamma}_{(l)}\}$. Since K_{ij} refers to the number of edges connecting the i^{th} and i^{th} nodes of the Dynkin diagram, we have to find those groups that have diagrams containing two edges between adjacent nodes. Referring to a list of Dynkin diagrams the condition

$$\frac{\boldsymbol{\beta}\cdot\boldsymbol{\gamma}}{\beta^2} = 1, \quad \boldsymbol{\beta} \neq \boldsymbol{\gamma} \tag{100}$$

may only be satisfied for the groups $SO(2n+1)$, $Sp(n)$ and F_4. Otherwise, namely for $SU(n)$, $SO(2n)$ and the other exceptional groups, $su(2)_\beta$ is the unique algebra containing $u(1)_\beta$ and the $\mathcal{M}_\beta$'s transform as in Eq. (94).

So what happens in $SO(2n+1), Sp(n)$ and F_4? We infer that it find that it is possible to perform a gauge transformation of a fundamental monopole into a state that has the same magnetic charge as a topologically distinct monopole. The monopoles can have different embeddings and different homotopy classes but still have the same magnetic charge. In a sense this is a form of generation structure.

C. Orbits of Embedded Monopoles

Before continuing with our discussion of monopoles it will be necessary to to discuss the adjoint action of the residual symmetry group upon $\mathcal{M}$. Recall that the adjoint action defines a representation of the massive gauge bosons under the residual gauge symmetry. Generally this representation is reducible into the gauge families of Eq. (14)

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n. \tag{101}$$

Thus the adjoint action gives an irreducible representation of H upon each $\mathcal{M}_i$.

An associated feature of this decomposition is the collection of gauge orbits under the action of the residual symmetry upon elements of $\mathcal{M}_i$. Such orbits are characteristic of both the representation and the symmetry. In particular the orbit that contains $X \in \mathcal{M}_i$ is of the form

$$\mathrm{Ad}(H)X \cong \frac{H}{C(X)}. \tag{102}$$

Collectively these orbits interleave to fill $\mathcal{M}_i$.

However, upon recalling Eq. (84), it becomes clear that the gauge orbit obtained from the gauge boson generators *is precisely the same* as the gauge orbit obtained from the gauge transformations of embedded monopoles. More exactly, Eqs. (84) and (102) imply that the gauge orbit of embedded monopoles under the residual symmetry group is

$$M(su(2)_Q) \cong \mathrm{Ad}(H)X, \qquad X \in \mathcal{M}_Q. \tag{103}$$

This is main result of this paper.

The implication of Eq. (103) is that associated with each monopole embedding is a massive gauge boson generator

$$su(2)_Q \leftrightarrow X \in \mathcal{M}_Q \tag{104}$$

such that the gauge orbits for both of these are exactly the same. Thus the element X can be though of as labelling the position of the $su(2)$ embedding within the gauge orbit. As the residual symmetry group moves the $su(2)$ embedding around it moves X around in a corresponding manner. This gives an equivalence between the gauge orbits of embedded monopoles and the gauge orbits of massive gauge bosons.

The above reasoning naturally leads to the following proposal: *Monopoles transform under the residual gauge symmetry, with an exact correspondence to the families of massive gauge bosons $\mathcal{M}_i$. Such gauge bosons lie in the adjoint representation of that symmetry.*

We discuss the implications of this proposal in the conclusion to this paper, and concentrate here on its details. In certain cases it is sufficient to consider only embedded monopoles, in which case the proposal has been rigourously proved. We discuss such cases at the end of this section, indicating firstly when the situation is more complicated.

In general there may also be non-embedded monopole solutions. Such solutions are not covered by our formalism, and generally relate to non-trivial combinations of fundamental monopoles. However, since we have explicitly displayed the gauge orbits for fundamental monopoles, this strongly indicates that the above proposal should hold generally. Indeed any symmetry of such non-embedded monopoles must be compatible with the symmetries displayed by the constituent fundamental monopoles.

An exact and explicit example of the above proposal occurs when $\mathrm{rank}(\mathcal{M}_i) = 1$. In that case the residual symmetry is transitive over generators $X \in \mathcal{M}_i$, and every embedded monopole corresponds to an associated generator $X \in \mathcal{M}_i$. Then it is not necessary to consider non-embedded monopoles since every gauge boson is in correspondance with an embedded monopole.

IV. EXAMPLE: $SU(5) \to SU(3) \times SU(2) \times U(1)/\mathbf{Z}_6$

The monopoles of Georgi-Glashow $SU(5)$ symmetry breaking are those of the dual standard model [9], where the gauge orbits of monopoles have been discussed at length [10]. We summarise some of those results here, using that work as an example of the formalism in this paper. For brevity we refer to $su(3) \oplus su(2) \oplus u(1)$ as $\mathcal{H}$.

Associated with the $SU(5)$ symmetry breaking is a corresponding split of the gauge boson generators into massive and massless families

$$su(5) = [su(3) \oplus su(2) \oplus u(1)] \oplus \mathcal{M}, \tag{105}$$

whose form may be represented as

$$\begin{pmatrix} su(3) & \vdots & \mathcal{M} \\ \cdots & \cdots & \cdots \\ \mathcal{M} & \vdots & su(2) \end{pmatrix} \times u(1) \subset su(5), \tag{106}$$

where $u(1)$ is along the diagonal, commuting with both $su(3)$ and $su(2)$. In regard to the gauge families $\mathcal{M}$ is irreducible under the adjoint action of $SU(3) \times SU(2) \times U(1)/\mathbf{Z}_6$. Therefore the residual symmetry group defines an irreducible representation upon $\mathcal{M}$.

Then we define a Cartan subalgebra $\mathcal{T}$ of the residual symmetry group. For consistency with the embedding in Eq. (106) we take the following diagonal generators,

$$T_1 = i \,\mathrm{diag}(-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{2}{3}, 0, 0), \tag{107}$$
$$T_3 = i \,\mathrm{diag}(1, -1, 0, 0, 0), \tag{108}$$
$$T_3 = i \,\mathrm{diag}(0, 0, 0, 1, -1), \tag{109}$$
$$T_4 = i \,\mathrm{diag}(1, 1, 1, -\tfrac{3}{2}, -\tfrac{3}{2}). \tag{110}$$

To provide a comparison with [10] we have not normalised these generators. This will effect the definition of the dual roots.

With respect to these generators the four roots of $\mathcal{H}$

$$\Phi(\mathcal{H}) = \{\pm\boldsymbol{\alpha}_{(0)}, \cdots, \pm\boldsymbol{\alpha}_{(3)}\}, \tag{111}$$

take the form

$$\boldsymbol{\alpha}_{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{112}$$

$$\boldsymbol{\alpha}_{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{\alpha}_{(2)} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{\alpha}_{(3)} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}. \tag{113}$$

In parallel with Eq. (105) the roots of $su(5)$ decompose into two sets

$$\Phi(su(5)) = \Phi(\mathcal{H}) + \Phi(\mathcal{M}). \tag{114}$$

Then the roots associated with massive gauge bosons

$$\Phi(\mathcal{M}) = \{\pm\boldsymbol{\beta}_{(1)}, \cdots, \pm\boldsymbol{\beta}_{(6)}\}, \tag{115}$$

take the form

$$\boldsymbol{\beta}_{(1)} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \quad \boldsymbol{\beta}_{(2)} = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \quad \boldsymbol{\beta}_{(3)} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \tag{116}$$

$$\boldsymbol{\beta}_{(4)} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \quad \boldsymbol{\beta}_{(5)} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \quad \boldsymbol{\beta}_{(6)} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}. \tag{117}$$

Each of the above roots $\boldsymbol{\beta}_{(i)} \in \Phi(\mathcal{M})$ has an associated $su(2)_{\beta_{(i)}}$ algebra embedded within $su(5)$ in the following manner

$$\begin{array}{ccc} su(5) & \to [su(3) \oplus su(2) \oplus u(1)] \oplus \mathcal{M} \\ \cup & \cup \qquad\qquad\qquad \cup \\ su(2)_{\beta_{(i)}} & \to \;\; u(1)_{\beta_{(i)}} = \mathbf{R}\boldsymbol{\beta}_{(i)} \cdot \boldsymbol{T} \;\; \oplus \mathcal{M}_{\beta_{(i)}}. \end{array} \tag{118}$$

The explicit embeddings are best pictured through the following diagram

$$\begin{pmatrix} & & & \vdots & \mathcal{M}_{\beta_{(1)}} & \mathcal{M}_{\beta_{(4)}} \\ & su(3) & & \vdots & \mathcal{M}_{\beta_{(2)}} & \mathcal{M}_{\beta_{(5)}} \\ & & & \vdots & \mathcal{M}_{\beta_{(3)}} & \mathcal{M}_{\beta_{(6)}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{M}_{\beta_{(1)}} & \mathcal{M}_{\beta_{(2)}} & \mathcal{M}_{\beta_{(3)}} & \vdots & su(2) & \\ \mathcal{M}_{\beta_{(4)}} & \mathcal{M}_{\beta_{(5)}} & \mathcal{M}_{\beta_{(6)}} & \vdots & & \end{pmatrix} \tag{119}$$

The above describes all of the structure required to specify the set of massive fundamental monopoles. These are 't Hooft-Polyakov monopoles defined on the subtheories $su(2)_{\beta_{(i)}} \to u(1)_{\beta_{(i)}}$. Since all such embeddings satisfy the condition $\mathcal{M}_{\beta_{(i)}} \subset \mathcal{M}$, they are embedded monopoles and therefore constitute solutions to the field equations.

In addition there are also embedded monopoles that are not fundamental. These are described in sec. (II D) and have embeddings diagonal within $\mathcal{M}_{\beta_{(i)}}$ and $\mathcal{M}_{\beta_{(j)}}$ such that $[\mathcal{M}_{\beta_{(i)}}, \mathcal{M}_{\beta_{(j)}}] = 0$. Within $su(5)$ the particular embeddings correspond to magnetic charge values $\{\boldsymbol{\beta}_{(i)} \pm \boldsymbol{\beta}_{(j-3\neq i)} : i \in \{1,2,3\}, j \in \{4,5,6\}\}$. Associated with each such magnetic charge is a non-fundamental embedded monopole.

The above embedded monopoles transform under a rigid gauge rotation of their embedding

$$su(2)_Q \mapsto \mathrm{Ad}(h) su(2)_Q, \tag{120}$$

with h an element of the residual symmetry group. The gauge orbit of all possible embeddings defines a manifold

$$\mathrm{Ad}(H) su(2)_Q \cong \mathrm{Ad}(H) X, \qquad X \in \mathcal{M}_Q. \tag{121}$$

In particular for the fundamental monopoles

$$\mathrm{Ad}(H) su(2)_{\beta_{(i)}} \cong \frac{SU(3) \times SU(2) \times U(1)/\mathbf{Z}_6}{SU(2) \times U(1) \times U(1)/\mathbf{Z}_2}. \tag{122}$$

In [10] this manifold was identified with the gauge orbit of the fundamental representation of H upon $\mathbf{C}^{3\times 2}$. This may be seen more easily with respect to Eq. (121), where the action of H upon the monopole embedding is given by the adjoint action of H upon $\mathcal{M}$. Since the adjoint action of H upon $\mathcal{M}$ is irreducible and $\mathcal{M} \cong \mathbf{C}^{3\times 2}$ we can infer directly that this action is the fundamental representation of H.

Within the gauge orbit of fundamental monopoles, given by Eq. (122), there are discrete points correspond to the roots $\beta_{(i)}$. The action of the residual symmetry group between such states is constructed from the Weyl subgroup W, consisting of the elements

$$\{\mathbf{1}, \exp[i\pi(E_{\alpha_{(a)}} + E_{-\alpha_{(a)}})/\sqrt{2\alpha_{(a)}^2}] : a \in \{0, 3\}\}. \tag{123}$$

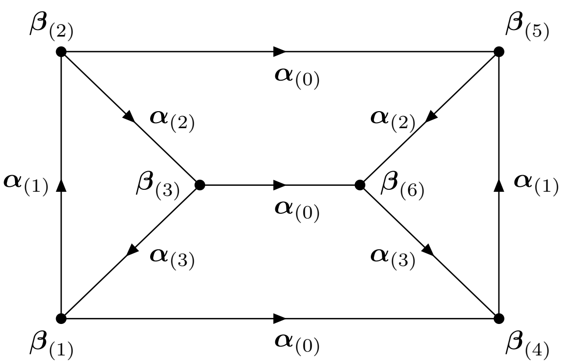
The action of $w_{\alpha_{(a)}} \in W$ is to transform

$$\mathrm{Ad}(w_{\alpha_{(a)}}) su(2)_{\beta_{(i)}} = su(2)_{\beta_{(j)}}, \tag{124}$$

where

$$\boldsymbol{\beta}_{(j)} = \sigma_{\alpha_{(a)}}(\boldsymbol{\beta}_{(i)}) = \boldsymbol{\beta}_{(i)} - 2\boldsymbol{\alpha}_{(a)} \cdot \boldsymbol{\beta}_{(i)} / {\boldsymbol{\alpha}_{(a)}}^2, \tag{125}$$

represents a Weyl reflection of the root $\boldsymbol{\beta}_{(i)}$ in the hyperplane $\boldsymbol{\alpha}_{(a)} \cdot \boldsymbol{x} = 0$. The effect of these may be summarised by the following diagram



V. DISCUSSION

In this discussion we tie together the different sections of this paper. Our aim is to show a similarity between the transformations of elementary particles and the transformations of monopoles under the residual symmetry group. To achieve this we shall highlight some features of this paper.

(1) Firstly, we are dealing with a spontaneously broken gauge theory $G \to H$, with certain restrictions on G and

H outlined at the beginning of this paper. The generators of the gauge bosons split into massless and massive families

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}. \tag{126}$$

The massive gauge bosons form a representation of the residual symmetry group H under the adjoint representation. This representation is reducible into irreducible parts

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n. \tag{127}$$

Each part represents a gauge family of massive gauge bosons.
(2) Secondly, embedded monopoles are $su(2)$ 't Hooft-Polyakov monopoles embedded in the larger gauge theory

$$\begin{array}{ccc} \mathcal{G} & \rightarrow & \mathcal{H} \\ \cup & & \cup \\ su(2)_Q & \rightarrow & u(1)_Q. \end{array} \tag{128}$$

Given a normalised set of commuting generators $\{T^i\}$ in $\mathcal{H}$, the massive fundamental monopoles are specified by the roots $\boldsymbol{\beta} \in \Phi(\mathcal{M})$, where $\Phi(\mathcal{G}) = \Phi(\mathcal{H}) + \Phi(\mathcal{M})$, such that

$$u(1)_Q = \mathbf{R}\,\boldsymbol{\beta} \cdot \boldsymbol{T}. \tag{129}$$

Writing $su(2)_Q = u(1)_Q \oplus \mathcal{M}_Q$ associates the embeddings with a decomposition of $\mathcal{M}_i$

$$\mathcal{M}_i = \sum_{\beta \in \Phi(\mathcal{M}_i)} \mathcal{M}_\beta. \tag{130}$$

In total $\Phi(\mathcal{M}) = \Phi(\mathcal{M}_1) + \cdots + \Phi(\mathcal{M}_n)$.
(3) Thirdly, we have shown that the action of the residual symmetry group H upon the $su(2)$ embedding of an embedded monopole is to take

$$su(2)_Q \rightarrow \mathrm{Ad}(h)\, su(2)_Q, \qquad h \in H. \tag{131}$$

(4) Finally, this action gives rise to a class of gauge equivalent monopoles. The features of this class may be described through the gauge orbit

$$M(su(2)_Q) \cong \frac{H}{C(su(2)_Q)}, \tag{132}$$

where $C(su(2)_Q)$ is the stabiliser of the monopole embedding

$$\begin{aligned} &C(su(2)_Q) = \\ &\quad \{h \in H : \mathrm{Ad}(h)X = X, \text{ for all } X \in su(2)_Q\}. \end{aligned} \tag{133}$$

The crucial feature of this gauge orbit is that

$$M(su(2)_Q) \cong \mathrm{Ad}(H)X, \quad X \in \mathcal{M}_Q. \tag{134}$$

The implication is that the action of H upon the embedding of a monopole is entirely equivalent to the action of H upon the massive gauge boson generated by $X \in \mathcal{M}_Q$.

Hence we infer that associated with each embedding of an $su(2)_Q$ monopole is an element $X \in \mathcal{M}_Q$. This element labels the embedding of the monopole. As the embedding is moved around its gauge orbit by the action of H the label X moves around an equivalent gauge orbit $\mathrm{Ad}(H)X$. In terms of acting the residual symmetry upon a monopole the concepts of embedding $su(2)_Q \subset G$ and considering an element $X \in \mathcal{M}_Q$ are completely interchangeable.

In keeping with the above reasoning we make the following proposal, proved for $\mathrm{rank}(\mathcal{M}_i) = 1$ and well motivated otherwise: *Monopoles transform under the residual symmetry, with an exact correspondence to the families of massive gauge bosons $\mathcal{M}_i$. Such gauge bosons lie in the adjoint representation of that symmetry.*

In other words the monopole embeddings are labelled by elements $X \in \mathcal{M}_i$. As such the embeddings appear to be internal degrees of freedom. This degree of freedom transforms exactly in keeping with a representation of the residual symmetry, in that a monopole embedding $X \in \mathcal{M}$ transforms as $X \mapsto \mathrm{Ad}(h)X$.

This behaviour is precisely the same as for elementary particles. Elementary particles are a collection of their representations. Their representation under Lorentz symmetry defines their spin. Their representation under gauge symmetry defines their interactions. The above results exhibit a similar nature for monopoles, with their embeddings defining their gauge representation. The behaviour of monopoles under spatial rotations has been similarly considered [11].

For illustration, consider a quark q transforming under colour $SU(3)_{\mathrm{C}}$ gauge symmetry. The quark field $\psi(\boldsymbol{r})$ takes values in $\mathbf{C}^3$ with the red, green and blue quarks forming a basis. This quark field naturally decomposes into a real magnitude and an internal direction

$$q(\boldsymbol{r}) = \psi(\boldsymbol{r}) v(\boldsymbol{r}), \quad v^\dagger v = 1, \psi \in \mathbf{R}, \tag{135}$$

whereby ψ represents the field theoretic nature of the quark, and v represents the internal degree of freedom. A gauge transformation of the quark field acts only upon the internal degree of freedom

$$v(\boldsymbol{r}) \mapsto v'(\boldsymbol{r}) = g \cdot v(\boldsymbol{r}), \quad g \in SU(3)_{\mathrm{C}}. \tag{136}$$

At a point $\boldsymbol{r}_0$ these maps out a gauge orbit

$$SU(3)_{\mathrm{C}} \cdot v(\boldsymbol{r}_0) \cong \frac{SU(3)_{\mathrm{C}}}{C(v(\boldsymbol{r}_0))}, \tag{137}$$

where $C(v(\boldsymbol{r}_0)) = SU(2) \subset SU(3)_{\mathrm{C}}$. This orbit is characteristic of both the quark's symmetry and representation.

Comparing the behaviour of the quark to the above properties of non-Abelian monopoles reveals that the label $X \in \mathcal{M}_i$ achieves exactly the same purpose as the internal degree of freedom for the quark. They take the same roles. Both define orbits that are characteristic of their gauge transformation properties. Both orbits are consistent with a representation of the gauge symmetry.

Additionally the red, green and blue quarks represent distinct points in the gauge orbit, with their span covering the space of coloured quarks. Fundamental monopoles play a similar role, with their span defining the lattice of magnetic charges.

We now enquire as to how a quantum theory of monopoles should effect the above picture. Although the treatment in this paper is semi-classical in nature, can the analogy with the above quark be carried any further? In particular the symmetries of a semi-classical monopole should naturally have a counterpart on the quantum level.

Technically, a representation and its linear structure is intrinsically linked to first quantisation. Semi-classical monopoles have a gauge orbit associated with their degeneracy. A field $\psi(\boldsymbol{r})$ then provides the linear structure that complements the gauge orbit to form a representation. Thus a statement that monopoles form a representation refers to the field theory of monopoles.

Monopoles become quantum mechanical when the Compton wavelength of a monopole is much smaller than its core. Estimating the Compton wavelength from the monopole mass

$$\lambda = m^{-1} \sim \frac{e^2}{4\pi}(ev)^{-1}, \tag{138}$$

this compares to the core size

$$r \sim (ev)^{-1}. \tag{139}$$

Thus the monopole is quantum mechanical when

$$\frac{e^2}{4\pi} \gg 1, \tag{140}$$

i.e. in the strong coupling regime of the gauge theory. In this regime it is entirely reasonable that the monopole would be represented by fields as in Eq. (135). Likewise, the gauge symmetry structure should carry over to that regime, as in elementary particles.

So, how do the interactions between monopoles behave in such a strong coupling regime? The magnetic charge of a monopole can be defined by integrating its asymptotic magnetic flux

$$Q_M = v^{-1} \int_{S_\infty} \mathrm{d}\boldsymbol{S} \cdot \langle \Phi, \boldsymbol{B} \rangle. \tag{141}$$

For the fundamental monopoles this is readily evaluated to be

$$Q_M = \frac{4\pi}{e}. \tag{142}$$

This indicates that the stregth of the inter-monopole interactions is proportional to the inverse of the gauge coupling. Therefore in the strong coupling regime the inter-monopole gauge interactions become well defined and perturbative in nature.

Such reasoning leads to the following picture. Whilst the gauge coupling is small, monopoles are semi-classical with a core size much larger than their Compton wavelength. They have a degeneracy given by a representation of the residual symmetry group, and these represent distinct gauge equivalent semi-classical monopole configurations. However, it is unclear how this degeneracy manifests itself in the dynamics as the associated magnetic coupling $4\pi/e$ is large.

When the coupling e is large the converse picture arises. The monopole is quantum mechanical, with a core much smaller than its Compton wavelength. Then the monopole degeneracy is fully compatible with a gauge interaction between monopoles; the gauge coupling $4\pi/e$ is small and the interactions are perturbative. In that case it is reasonable to view the concept of an individual monopole as misleading and instead think of the monopoles as a quantum field. Such a field should be compatible with the underlying symmetries of the monopole configuration.

In conclusion we reiterate the results of this paper. We have shown that the gauge orbits of semi-classical embedded monopoles are fully consistent with a representation of the residual gauge symmetry. In particular this representation is that of the residual gauge symmetry upon the massive gauge bosons. In the strong coupling regime these results are consistent with monopoles assuming the role of elementary particles that interact perturbatively under the residual gauge symmetry.

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APPENDIX A: EMBEDDING MONOPOLES

For the analysis of this paper we require that embedded monopoles are defined by the inclusion $\mathcal{M}_Q \subseteq \mathcal{M}_i$, where $su(2)_Q = u(1)_Q \oplus \mathcal{M}_Q$. We now prove that this condition necessarily and sufficiently defines the embedded monopoles.

It will be necessary to use the field equations of the Lagrangian in Eq. (7). These are

$$D^\mu D_\mu \Phi = -\frac{\partial V}{\partial \Phi}, \tag{A1}$$

$$\langle D^\mu F_{\mu\nu}, X \rangle = \langle J_\nu, X \rangle \tag{A2}$$

$$= \langle \mathrm{ad}(X)\Phi, D_\nu \Phi \rangle - \langle D_\nu \Phi, \mathrm{ad}(X)\Phi \rangle, \tag{A3}$$

where the scalar field takes values $\Phi \in \mathcal{V} = \mathcal{G}$ and the gauge field takes values $A^\mu \in \mathcal{G}$.

An embedded monopoles is a non-Abelian 't Hooft-Polyakov monopole Ansatz

$$\Phi(\boldsymbol{r}) = vh(r)\,\mathrm{Ad}(g(\Omega))\Phi_0, \tag{A4}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1-u(r)}{er}\,\hat{\boldsymbol{r}} \wedge \boldsymbol{t}_Q, \tag{A5}$$

$$g(\theta,\varphi) = \exp(\theta(t^2\sin\varphi - t^1\cos\varphi)). \tag{A6}$$

defined on the embedded theory $su(2)_Q \subset \mathcal{G}$. This embedding naturally decomposes $\mathcal{G}$ globally into

$$\mathcal{G} = su(2)_Q \oplus su(2)_Q^\perp. \tag{A7}$$

Likewise it decomposes the scalar field values into

$$\mathcal{V} = \mathcal{V}_Q \oplus \mathcal{V}_Q^\perp, \tag{A8}$$

where

$$\mathcal{V}_Q = \mathbf{R}\,\Phi_0 \oplus \mathbf{R}\,\mathrm{ad}(\mathcal{M})\Phi_0. \tag{A9}$$

The above Ansatz provides a solution providing that fields in the embedding do not source fields outside that embedding. This requires

$$\langle D^\mu D_\mu \Phi(\boldsymbol{r}), \mathcal{V}_Q^\perp\rangle = 0, \tag{A10}$$

$$\langle J^\mu(\boldsymbol{r}), su(2)_Q^\perp\rangle = 0. \tag{A11}$$

Application of the field equations then implies

$$\langle \Psi, \frac{\partial V}{\partial \Phi}[\phi]\rangle = 0, \quad \Psi \in \mathcal{V}_Q^\perp, \phi \in \mathcal{V}_Q, \tag{A12}$$

$$\langle \mathrm{ad}(Y)\Phi, \mathcal{V}_Q\rangle = 0, \quad \Phi \in \mathcal{V}_Q, Y \in su(2)_Q^\perp. \tag{A13}$$

Since Eq. (A7) is invariant under the action of $su(2)_Q$ and the inner product is invariant under the action of G, Eq. (A13) may be expressed equivalently as

$$\langle \mathrm{ad}(Y)\Phi_0, \mathcal{V}_Q\rangle = 0. \tag{A14}$$

Then using Eq. (A9) and $\langle \mathrm{ad}(\mathcal{G})\Phi, \Phi\rangle = 0$ expresses this as

$$\langle \mathrm{ad}(Y)\Phi_0, \mathrm{ad}(\mathcal{M}_Q)\Phi_0\rangle = 0, \quad Y \in su(2)_Q^\perp. \tag{A15}$$

To this we apply the following result, proved in [8]:

$$\langle \mathrm{ad}(X_i)\Phi_0, \mathrm{ad}(Y_j)\Phi_0\rangle = \lambda_i\lambda_j\langle X_i, Y_j\rangle, \qquad X_i \in \mathcal{M}_i, Y_j \in \mathcal{M}_j, \tag{A16}$$

with $\lambda_i = \parallel \mathrm{ad}(X_i)\Phi_0 \parallel / \parallel X_i \parallel$. This shows that the orthogonality between $su(2)_Q$ and $su(2)_Q^\perp$ is respected only for embeddings within the gauge families. Hence we infer that the monopole embeddings are specified by the $\mathcal{M}_Q$ that are defined by

$$\mathcal{M}_Q \subseteq \mathcal{M}_i. \tag{A17}$$

Completing the proof.

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